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Conceptual Understanding of Dot Product of Vectors in a Dynamic Geometry Environment

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Abstract

Current trends in research on the impact of technologies in mathematics education emphasize their increased role in supporting students' conceptual understanding in comparison with numerous previous studies about technology contribution in procedural understanding. This article exemplifies the role of Dynamic Geometry Systems utilizing students' understanding of concepts in Linear algebra in the transition between upper high school and university education. Students' conceptual understanding is identified through guiding features as: concept definitions and concept images, according to the theory of Tall and Vinner (1981); three modes of description and thinking: arithmetic, geometric and axiomatic-structural, according to the theories of Hillel (2000) and Sierpinska (2000); concepts' properties which construct axiomatic definitions; and concepts' applications and their connections with other concepts. Authentic video recordings serve as a collected data set for qualitative analysis of students' interactions in the designed Dynamic Geometry Environment. The study is part of a larger design-based research undergoing seven phases in a cyclic manner, ending with evaluation and dissemination of created teaching and learning materials as visual dynamic applets and worksheets.

1 Introduction

This article is consisted of nine sections. After the introduction, the second section examines different theories which define the term "conceptual understanding" and offers a possible identification of this elusive term through five guiding features and examples in Linear algebra. The third section represents an overview on the current research trends of the utilization of technologies, in particular Dynamic Geometry Systems (DGS), in supporting students' conceptual understanding. After that, in the fourth section, a particular attention is set on one of the guiding features, namely, the three different modes of description and thinking in Linear algebra. Then, dot product of vectors is chosen as an exemplary concept for illustrating the previously discussed modes of description and thinking. The following section clearly states the research questions and the chosen research methodology. The article continues with a suggestion of a possible design which could support conceptual understanding of the chosen concepts through all three modes of description and thinking. The third mode, the axiomatic-structural mode, stays in the focus in this section, because it strengthens the bridge between the upper high school and university level of Linear algebra. The design uses dynamic applets created in GeoGebra. Practical implementation of the suggested design in school is elaborated in the eight section. The article ends with a section for conclusions and discussion.

2 What is Conceptual Understanding?

The problem of an exact definition of conceptual understanding of mathematical concepts remains an important topic in the research debate. Conceptual understanding could be perceived as a structure or a network of mathematical ideas or representations in which the degree of understanding depends on the number and strength of its connections [13, p. 67]. Similar to this view is that conceptual understanding could be defined as the resulting network consisted of connections between mental representations of a mathematical concept, thus an action and a result of an action [5]. In comparison with the previous one, this view gives an additional attribute of conceptual understanding representing a process, not only a static network, but a network with dynamical characteristics. Other theories distinguish between more types of understanding, namely: rational and instrumental types of understanding [23] or operational and structural [20] or procedural and conceptual understanding [13]; [12]. Widely accepted theories, such as Piaget's and APOS (action-process-object-scheme) theories, from the International Group for the Psychology of Mathematics Education consider conceptual understanding as a permanent growth which overcomes several phases. Compared to these last two theories, Bruners E-I-S (enactive-iconic-symbolic) theory represents a global framework of permanent conceptual growth [6]. Tall's theory builds on the foundations of the Bruners E-I-S trio and supplements it by a fourth, formal axiomatic level in the conceptual development [26]. These four phases in the long lasting conceptual growth spread over more periods of educational processes: from the lower secondary- through the upper secondary- to the university- and further education. A cohesion of the Bruner's and Tall's theories is accepted as a suitable theoretical framework for this study with an emphasis that strong borders between any two of the mentioned phases of conceptual growth do not exist. On the contrary, development of one of them may strongly influence stimulation of another one. In this sense, learning about the concept of dot product of vectors exemplifies such conceptual development and therefore it is taken as a central mathematical concept in this study. It is considered as an important concept at university Linear algebra and Calculus of the transition between upper high school and university could successfully be described through the spiral metaphor of conceptual growth of the theory of Bruner (1966) and the work of Tall (2004).

2.1 Five Guiding Features of Conceptual Understanding. Examples in Linear Algebra

A brief summary of the above theories, in particular Bruner's (1966) and Tall's (2004) theories, leads to stating five features of conceptual understanding in Linear algebra. These five features are illustrated by an exemplary concept, namely dot product of vectors. They are considered relevant in this study and used for data analysis, with an accent on the third feature. Namely, students are in an *enactive* phase when they come up with heuristics as a result of their engagement in pragmatic- and epistemic-valued activities in a technology-enhanced environment (explained at the end of Section 7 in this article). This also refers to the *action* phase in the APOS theory, or what students do in the learning environment. The *iconic* phase can be related to a *geometric mode of description*, i.e. geometric figures representing a particular Linear algebra concept (for example, oriented areas of rectangles representing dot product of vectors). The *symbolic* phase can be connected to an *algebraic mode of description*, when mathematical symbols are used for representation or calculation of the value of the concept in concern (in this case, the dot product of vectors). In the same time, the iconic representation can be associated to a geometric image and the symbolic representation to an algebraic image of the same concept. Then, Tall's *axiomatic* phase builds on the bases of these phases integrating concept's properties in a single concept's definition. Such an axiomatic definition represents an *abstract-structural mode of description* of the concept.

For easier tracking of these five features of conceptual understanding they are listed as follows:

1. *Distinguishing what is and what is not a dot product of vectors?* Examples and counter examples for a particular concept, in this case dot product of vectors; make a significant difference between crucial features of the concept which may be overlooked if not pointed out explicitly.

   Example 1: Dot product of vectors is a scalar and not a vector.

2. *Concept definitions and concept images* [24] of dot product of vectors. Existence of more possible definitions for a single concept is often not perceived by students. Even if they are aware of the existence of multiple concept definitions, the problem of establishing links between them remains discussible. Thus, what kind of concept images do students form for the concept of dot product of vectors in particular, is an interesting phenomenon to be observed in this study.

   Example 2a: Dot product of vectors is the product of vectors magnitudes and the cosine of the angle between them.

   Example 2b: Dot product of vectors is the sum of the products of corresponding vectors components.

3. *Multiple modes of description, language and thinking* [14]; [22] of dot product of vectors. There exist multiple modes of description, language and thinking of dot product of vectors and an investigation of how do students recognize, translate among and manipulate different modes is of particular interest in this study. These modes are discussed in section 4 and section 5 of this article.

   Example 3: Definition given in example 2a utilizes geometric mode of description and thinking of dot product of vectors, while the definition in example 2b utilizes arithmetic mode of description of dot product of vectors.

4. *Concept's properties which construct an axiomatic definition of the dot product of vectors, the existence and uniqueness conditions in such definition.* Questions whether axiomatic definitions should be part of the upper high-school curricula for Linear algebra seem to have been abandoned after their overuse within the New Math. It appears today, that with an adequate approach, possibly supported by technology, they may be brought into contexts which are accessible for high-school students to a certain degree (without proofs for the existence and uniqueness conditions).

Example 4: Dynamic visualization, including geometric and arithmetic-algebraic modes of descriptions, of axiomatic properties of dot product of vectors as: bi-linearity (homogeneity and additive properties), symmetric and positive properties are suggested in the proposed design of this study in section 7.

5. *Connections of the concept dot product of vectors with other concepts.* Forming a structured network between the concept of concern and other concepts within Linear algebra and analytic geometric, and also concepts out of this field, that supports students conceptual understanding. Concept's applications in problem solving situations, proving or modeling also support conceptual understanding.

   Example 5: Connections between dot product of vectors and the trigonometric function cosine of an angle contribute to constructions of a wider network between Linear algebra and Trigonometry.

3 The Role of Technology in Supporting Students' Conceptual Understanding in Linear Algebra

The scientific debate on the technology use in mathematics education never seems to end. In the last three decades a huge progress in research has been made, but are the results of the practical implementations of different studies satisfactory, remains a question. There is evidence for a discrepancy between the great amount of literature offering a wide range of theories, methodologies and interpretations of the ICT integration in mathematics education in schools [15].

> After more than two decades of concerted effort, less than desirable progress has been made in the integration of effective technology use in the classroom. In many cases computers have entered classrooms, but are used to do little more than support existing teaching practices i.e. PowerPoint read as lecture [11, p. 57].

Technology usage in mathematics education certainly has much bigger potentials than what stated in the above citation. There has been a shift from using computer technologies to promote drill-and-practice in traditional mathematics classrooms to using them to create an interactive learning environment in learner-centered mathematics classrooms [17].

> Interventions focusing on the development of conceptual understanding produced an average effect size almost double that of interventions focusing on procedural understanding [19, p. 372].

The role of technology utilization in this study is to show that the conventional idea, that axiomatic approaches are exclusively reserved for university Linear algebra, could be viewed from another perspective. Namely, it could be viewed in the context of the existence of multiple modes of description and thinking of a single concept and thus brought closer to upper high school mathematics. In more specific context, the study shows how technology, more specifically a dynamic geometry environment, facilitates conceptual development by bridging the gap between the concrete-symbolic and the formal-axiomatic worlds of the theory of [26].

4 Modes of Description and Though in Linear Algebra

Many researchers emphasize that proper combinations of representations lead to improved students' learning outcomes [2], translations between different representations support conceptual understanding [18] and are important for acquiring deeper knowledge about a domain [28]. It is well known that quick and correct calculations or apparently fluent procedural skills are not necessarily followed by conceptual understanding. Previous research reports that one of the indicators of conceptual understanding is the capability for recognizing structurally the same connections formed via multiple representations [18, p. 2]. Current research studies identify students' difficulties in recognizing multiple representations of a single concept in Linear algebra [7] and existence of limitations in students' understanding the variety of modes of description [10]. Dubinsky & Wilson (2013) report that even those who are able to recognize different modes of description cannot form links across them [10]. They state that the algebraic mode of description is preferable mode by many students and teachers and there is often an intention for substituting a particular mode of description with another one even in unnecessary situations [10]. Despite the contribution of these studies, students' experiences with multiple representations of Linear algebra concepts remain an undiscovered area [7] especially by means of technology. How can translations across more representations be supported to maximize students' learning outcomes and effectiveness of multiple-representational learning environments? The phenomenon of dynamic multiple representations in computer based learning environments in comparison with: single static representations, single dynamic representations and multiple static representations offers the most opportunities and challenges [28].

Let us first explain multiple representations of Linear algebra concepts, the specific terminology, meaning and usage in this article[1]. Hillel's (2000) theoretical framework encloses three modes of description and language of Linear algebra concepts:

- *geometric* mode of description and language,
- *algebraic* mode of description and language,
- *abstract* mode of description and language.

In close correlation and for the purpose of establishing connections between these three modes of description and language, [14] distinguishes between two modes of representations: geometric-algebraic mode and algebraic-abstract mode of representation. Upgrading this theoretical framework, Sierpinska and co-authors [22]; [9, p. 209]; [21] describe three modes of thoughts of Linear algebra concepts as follows:

- *Geometric language/ synthetic-geometric mode* of thought refers to 2- and 3- space (directed line segments, points, lines, planes, and geometric transformations).
- *Arithmetic language/ analytic-arithmetic mode* of thought refers to n-tuples, matrices, rank, solutions of systems of equations, etc.
- *Algebraic language/ analytic-structural mode* of thought refers to the general theory (vector spaces, subspaces, dimension, operators, kernels, etc.).

[1] The explanation offered here occurs also in [8, pp. 305-307].

A comparison between Hillel's (2000) modes of description and Sierpinska's (2000) modes of though shows a notable similarity. Namely, the Hillel's geometric mode of description is closely related to the Sierpinska's synthetic-geometric mode of thought; the algebraic mode of description to the analytic-arithmetic mode of thought; and the abstract mode of description to the analytic-structural mode of thought, respectively.

In continuation, the above stated modes of description and thinking are discussed in particular for the concept dot product of vectors.

5 Dot Product of Vectors

Dot product of vectors is an important Linear algebra concept. Introduction to this concept in upper secondary school is usually undertaken in either arithmetic or geometric mode of description. It is a rare case that both of these modes of defining the concept are simultaneously offered to the high school students. The third possible mode, the abstract-axiomatic mode, of description is usually reserved for university level of Linear algebra. Definitions of dot product of vectors in each of the three modes of description are given in the next three subsections.

5.1 Definition in Arithmetic Mode of Description and Language

The definition of dot product of vectors in the arithmetic mode of description is given through vectors' components, which in its nature distinguishes the dimension of space in which it is given.

Definition 1. For two vectors $\overrightarrow{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ given with their components in R^n, their dot product is the real number $\overrightarrow{u} \cdot \overrightarrow{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$.

The special cases of Definition 1 for vectors given in dimension $n = 2$ and $n = 3$ respectively follow.

For two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ given with their components on a plane, their dot product is the real number $\overrightarrow{u} \cdot \overrightarrow{v} = u_xv_x + u_yv_y$ [1, p. 114].

For two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ given with their components in space, their dot product is the real number $\overrightarrow{u} \cdot \overrightarrow{v} = u_xv_x + u_yv_y + u_zv_z$ [1, p. 114].

5.2 Definition in Geometric Mode of Description and Language

Synthetic-geometric mode of description and thought of dot product of vectors is a coordinate-free way to define the concept. It is as follows.

Definition 2. For two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ and the angle φ between them, their dot product is: $\overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}| \cdot |\overrightarrow{v}| cos\varphi$.

Definition 2'. For two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ their dot product is: $\overrightarrow{u} \cdot \overrightarrow{v} = \pm|\overrightarrow{u}||\overrightarrow{v}_{\overrightarrow{u}}| = \pm|\overrightarrow{v}||\overrightarrow{u}_{\overrightarrow{v}}|$, where $\overrightarrow{v}_{\overrightarrow{u}}$ is a projection of the vector $\overrightarrow{v}$ over the vector $\overrightarrow{u}$ and $\overrightarrow{u}_{\overrightarrow{v}}$ is a projection of $\overrightarrow{u}$ over $\overrightarrow{v}$.

Note. Definition 2 and Definition 2' are equivalent.

5.3 Definition in Abstract-Axiomatic Mode of Description and Language

Dot product of vectors in the abstract-axiomatic mode of description is defined through three axioms, i.e axioms for bilinearity (additive and homogeneity), symmetry and positivity.

Definition 3. The linear mapping $\cdot : R^n \times R^n \to R$, $(\overrightarrow{u}, \overrightarrow{v}) \mapsto \overrightarrow{u} \cdot \overrightarrow{v}$, with the properties:

1. Bi-linearity:

   1a. Scaling:
   $\lambda(\overrightarrow{u} \cdot \overrightarrow{v}) = (\lambda\overrightarrow{u}) \cdot \overrightarrow{v}$,
   $\lambda(\overrightarrow{u} \cdot \overrightarrow{v}) = \overrightarrow{u} \cdot (\lambda\overrightarrow{v})$,

   1b. Homogenity.
   $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}$,
   $(\overrightarrow{u} + \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{u} \cdot \overrightarrow{w} + \overrightarrow{v} \cdot \overrightarrow{w}$,

2. Symmetry: $\overrightarrow{u} \cdot \overrightarrow{v} = \overrightarrow{v} \cdot \overrightarrow{u}$,

3. Positivity: $\overrightarrow{u} \cdot \overrightarrow{u} \geqslant 0$ and $\overrightarrow{u} \cdot \overrightarrow{u} = 0 \Leftrightarrow \overrightarrow{u} = \overrightarrow{0}$,

where $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w} \in R^n$ and $\lambda \in R$, is called a dot product.

Here is a small comparison between the above definitions in the sense of the theories of Hillel (2000) and Sierpinska (2000) about different modes of description and language. Definition 1 is in algebraic mode of description [14], i. e. arithmetic language [22], because it utilizes n-tuples as vectors representations and defines the concept of dot product of vectors through numbers and number operations as addition and multiplication. Definitions 2 and 2' include terms as length, angle and projection and are therefore given in the geometric mode of description and language. With the Definition 3, the dot product of vectors is characterized through axioms. For this reason it is an example of a definition in abstract mode of description [14] and structural mode of thinking [22]. Existing correlation between these definitions given in different modes of description and thinking is the third guiding feature of conceptual understanding (in section 2.1).

6 Research Questions and Methodology

This section clearly states the research question, which specifies the direction of intended investigation within this study.

- *Research Question (RQ).* How does a dynamic geometry environment (DGE) support use of three modes of description and thinking specifically for dot product of vectors?

Having in mind the five guiding features of conceptual understanding elaborated in subsection 2.1, contributions of dynamic geometry systems in offering answers to a question as: how does a DGE support students' conceptual understanding of dot product of vectors, are broad. Therefore, the research focus in this study is set mainly on the third guiding feature about the three modes of description and thinking of dot product of vectors, as it is specified in the RQ, although the other features are mentioned when appropriate.

The appropriate methodology aiming to provide answers on these questions is a qualitative analysis of undertaken teaching/ learning experiments in an upper high-school with mathematics orientation in Berlin. Participants are 12th grade students and their mother-tongue language is German[2] (which can be noticed by reading the excerpts of the transcripts of the video recordings in section 8.

7 Suggested Design for Supporting Conceptual Understanding of Dot Product of Vectors in a DGE

Following the elaborations and guiding features about conceptual understanding of Linear algebra concepts in section 2, this section suggests a design in a dynamic geometry environment (DGE) to intercorrelate all three concept definitions of dot product of vectors given in section 5. Meanwhile, the created design also integrates all three modes of description and thinking of dot product of vectors, because both applets [30] and [31] (shown on Figure 1 and Figure 2) simultaneously contain arithmetic, algebraic and geometric parts which are synchronized under dragging, which refer to **Definition 1** and **Definition 2** in section 5. Moreover, they visualize corresponding axioms in the **Definition 3** in section 5.

Figures 1a) and 1b) below show the scaling property $\lambda(\vec{u}\cdot\vec{v}) = (\lambda\vec{u})\cdot\vec{v}$, for the vectors $\vec{u} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and $\lambda = 2$.

Figures 1c) and 1d) show the scaling property $\lambda(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(\lambda\vec{v})$ for the same vectors $\vec{u}$ and $\vec{v}$ and the same scalar $\lambda = 2$.

[2]The language of instruction and communication during the experiments in the school is not a matter of the analysis in this article, although it was taken into consideration within the whole study.

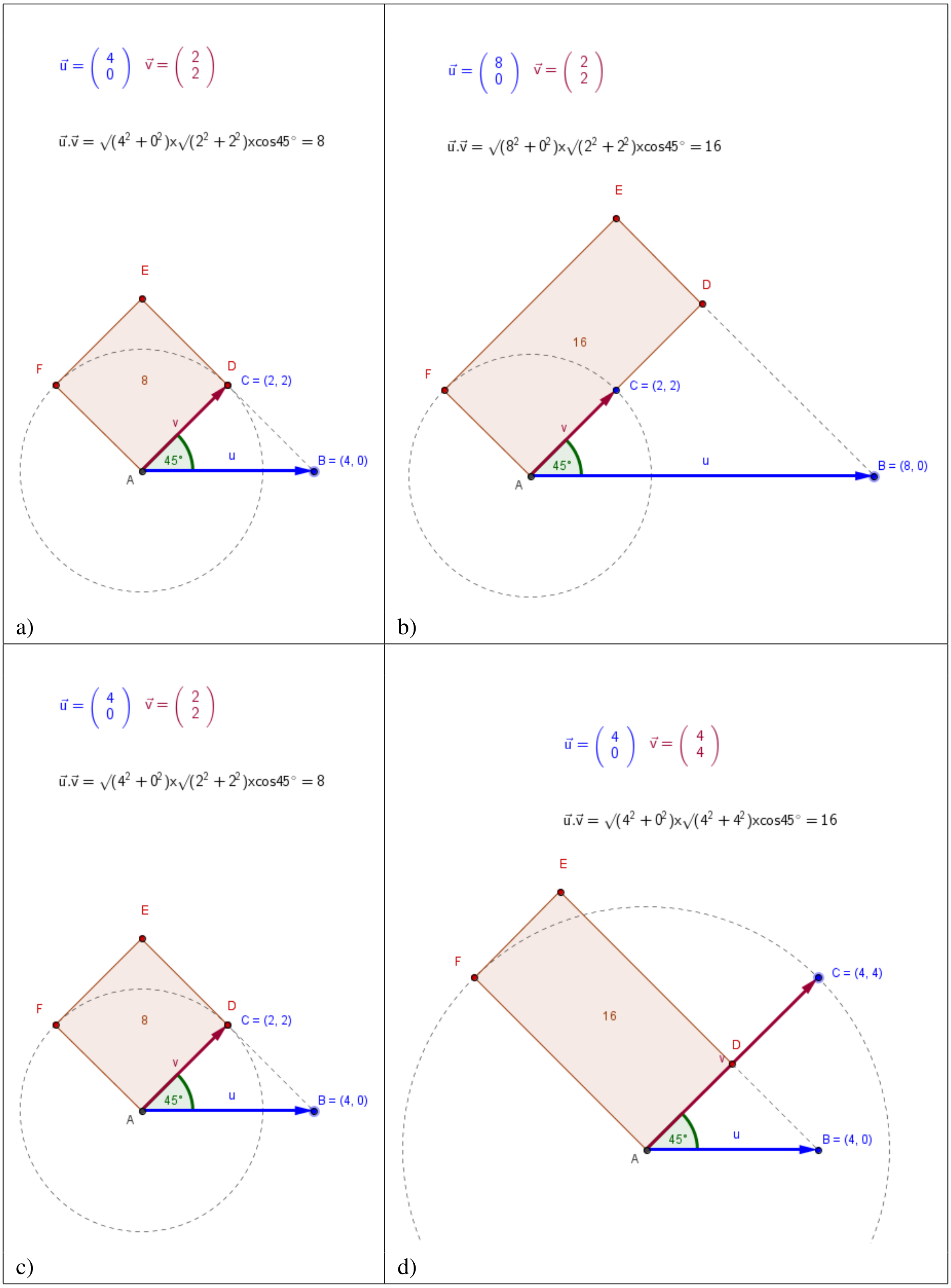

Figure 1: Axiom 1a. Scaling Property of Dot Product of Vectors

The dynamic visualization on the Figure 1 illustrates the geometric interpretation of dot product of vectors as an oriented area of the rectangle formed by the magnitude of one of the vectors, namely $\overrightarrow{v}$, and the magnitude of the projection of the other vector $\overrightarrow{u}$ over $\overrightarrow{v}$.

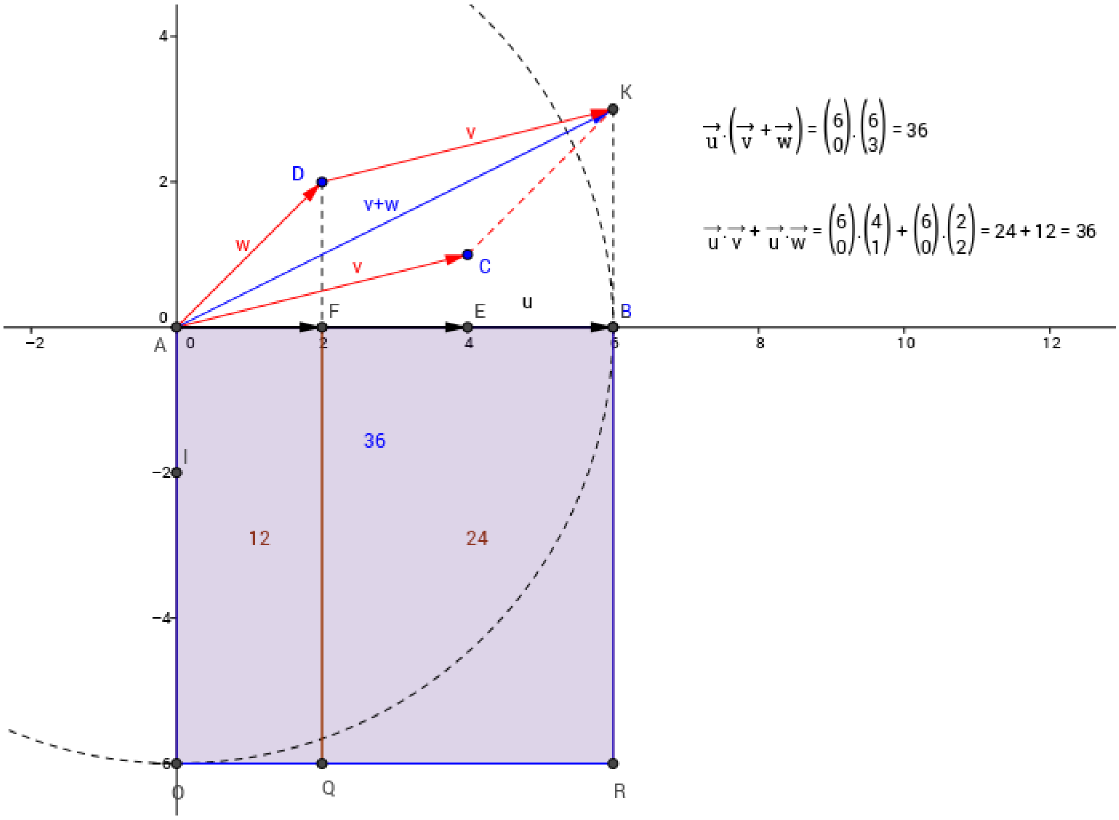

Figure 2: Axiom 1b. Homogeneity Property of Dot Product of Vectors

Figure 2 shows a position of the second applet created in the dynamic geometry environment for visualizing the Axiom 1b., $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}$, i.e. the homogeneity property of dot product of vectors.

Figure 3 visualizes the Axiom 3. $\overrightarrow{u} \cdot \overrightarrow{u} \geqslant 0$ and $\overrightarrow{u} \cdot \overrightarrow{u} = 0 \Leftrightarrow \overrightarrow{u} = \overrightarrow{0}$, for positivity of dot product of vectors.

Both applets (shown on Figures 1 and 3; and Figure 2), together with related questions, are enclosed within two corresponding interactive [GeoGebra] Worksheets [30] and [31] .

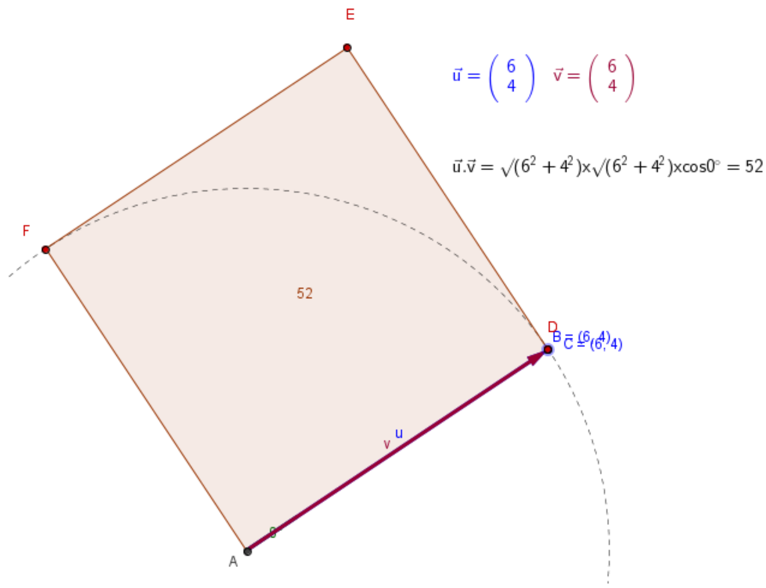

Figure 3: Axiom 3. Positivity of Dot Product of Vectors

When working in the designed DGE in this study, students practically follow a Variational Dragging Scheme (VDS) for the applet, which includes focus on simultaneous changes of:

- Coordinates of points
- Vectors components
- Vectors magnitudes and directions
- Vectors orientations (counter-clockwise or clockwise)
- Vectors projections
- Angles between vectors (affecting the signs or zero value of the dot product)
- Areas of corresponding rectangles

The dynamic characteristic of the environment offers students a possibility to discover how changes of points' coordinates (algebraic mode) influence areas of corresponding rectangles (geometric mode) which do not change in other quadrilaterals. Any choice of the vectors' components preserves the rectangular shape of the figures representing the dot product of those vectors. This is not easy to show by static images, but is quickly testable with the dragging tool in the created DGE. Such epistemic use of the pragmatic activity of dragging [3], [4] shows the importance of the technology-based design in a didactical context.

8 Practical Implementation in School and Didactical Considerations

This section shows how students interact with the applet [30] of the proposed design for supporting conceptual understanding of dot product of vectors in a DGE. By dragging modalities (Variational Dragging Schemes stated at the end of Section 7), changing lengths and directions of vectors and also angles between them they have to discover that dot product can be referred only to particular quadrilaterals, namely rectangles and squares and not other parallelograms. Furthermore they have to convince themselves that absolute value of dot product equals the area of the obtained rectangle (one side of the rectangle equals the length of one of the vectors and the other side equals the length of the projection of the other vector) by the previously learned definitions. Possible students' misconceptions regarding the term oriented area are prevented such that positive areas are displaced on the rectangles while $\pm$ values of the dot product appear in the arithmetic-algebraic mode of description on the top of the applet [30].

[1] Instructor: Do you know what a dot product of two vectors is?

[2] A couple of students: Yes [aloud in one voice].

[3] S1: So, it's a number!

[4] Students: [all laugh]

[5] Instructor: OK, so at least you know it's a number and not a vector. Which number exactly? How can we obtain this number? Maybe anyone could write it on the board?

[6] S2: [unclear vague explanation in German language, then writes definition of dot product in component form on the board, as **Definition 1** stated in Subsection 5.1 in this article].

[7] Instructor: OK, It's correct. Do you maybe know another definition of dot product of vectors? [addressing the question to the whole class].

[8] S3: [first writes cosine of the angle and then definition of dot product on the board, as **Definition 2** stated in Subsection 5.2 in this article, without any comments].

The beginning of the above transcript, when the concept of dot product is mentioned, shows students' sympathy, positive reactions and good working atmosphere (lines [2] to [4]). Similar positive students' emotions towards this concept have also been detected by [29, p. 288]. Lines [1] to [5] in the excerpt from the transcript show that students already know that dot product of vectors is scalar (S1 in line [3]) and not a vector, which meets the first guiding feature towards conceptual understanding which is distinction between what is and what is not a dot product of vectors (Subsection 2.1). Further on, students by themselves experience (S2 in line [6], and S3 in line [8]) that a concept may be given with more than one formulas, which is the second guiding feature towards conceptual understanding. Student's S2 writing on the board is actually the arithmetic-algebraic mode of description, while student's S3 writing is the geometric mode of description of dot product of vectors. More formulas may exist even for a single mode of thinking of a concept, for example the formulas $\overrightarrow{u} \cdot \overrightarrow{v} = |\overrightarrow{u}||\overrightarrow{v}|cos\varphi$ and $\overrightarrow{u} \cdot \overrightarrow{v} = \pm|\overrightarrow{u}||\overrightarrow{v}_{\overrightarrow{u}}|$, both referring to the synthetic geometric mode of description and thinking of the concept. They are, of course, equivalent, though it may not appear to the students on the first look. Noticing and understanding this equivalence is one of the 'tasks' of the applet [30].

What seem to be lacking is an oral explanation and understanding of the existing connections between the two written formulas for dot product of vectors. Namely, the student 3 first wrote the cosine of the angle and only after a request wrote the definition of the dot product of the vectors. This spontaneous student's reaction arrives from his previous exposure on the application of the dot product for measuring angles, thus confirms previous assumptions that the introduction of this concept in school is often limited to its application for measuring angles. Further on, students are able to write the geometric definition, but do not really understand what does it exactly mean. It seems that students are able to memorize these formulas, but their underlying understanding is symptomatic. Wittmann's case study about this concept shows the importance of memorizing formulas for solving exercises and exam problems for an interviewed student [29, pp. 220-221]. Memorizing formulas is definitely not all we want to teach our students. Can technology facilitate situations like this one and how? One way is through asking students for explanations in their own words. The discussion when technology comes on stage continues with the following excerpt of the transcripted video recording:

[9] Instructor: [...] Could you explain what do you see on this applet? [Figure 4.]

[10] S4: So, we have the $\overline{AF}$ with the same length as $\overline{AC}$, because C and F are both on the circle around A. So we ... [turning to a student] Was ist Rechtecke? [asking for the word "rectangle" in English].

[11] Instructor: Rectangle.

[12] S4: Rectangle $AFED$ with two sides $\overline{AC}$, so the length of the vector $\overrightarrow{AC}$ times the part of the vector $\overrightarrow{AB}$ which is ... ähm ... yeah, it's hard to explain [noticeable problem with his explanation into English language, lacking the word "projection"] which would shine directly onto $\overrightarrow{AC}$ [pointing on the screen] then the point D would be the shadow of the point B (Figure 4).

[13] Instructor: And how is that related to the definition?

[14] S4: Yeah, of course, here we have $90°$ [looking at the applet] so the cosine of the angle is this divided by this [pointing on the applet].

[15] Instructor: What do you mean "this dived by this"?

[16] S4: [laughing] of course yes. $\overline{AD}$ divided by $\overline{AB}$ ... is cosine of this angle between the vectors.

[17] Instructor: So in this way you find one of the sides of the rectangle and finally, what can you say about the area of the rectangle?

[18] S4: The area is the length of $\overrightarrow{AC}$ times the length of the $\overrightarrow{AB}$ times cosine of the angle, so therefore it is Skalarprodukt.[3]

[19] Instructor: Thank you!

On the first instructor's question [9] for an explanation of the applet, although without explicit use of the words "vector projection", i.e. correct terminology, the student S4 immediately recognized the projection and explained it in his own words with the phrases "[...] would shine directly onto [...]" and "the point D would be the shadow of the point B" in [12]. This coincides with the **Definition 2'** stated in the Subsection 5.2. This student's discovery shows that he undergoes the *enactive* and the *iconic* representations and is close to reaching the symbolic representation of the Bruner's trio, because not only he recognizes, but also verifies correct mathematical content articulating it using own words. Besides performing the VDS, the student additionally used his hands to enactively explain his eureka on the computer screen, students' actions described in research as an embodied world

[3] Skalarprodukt is the German word for dot product.

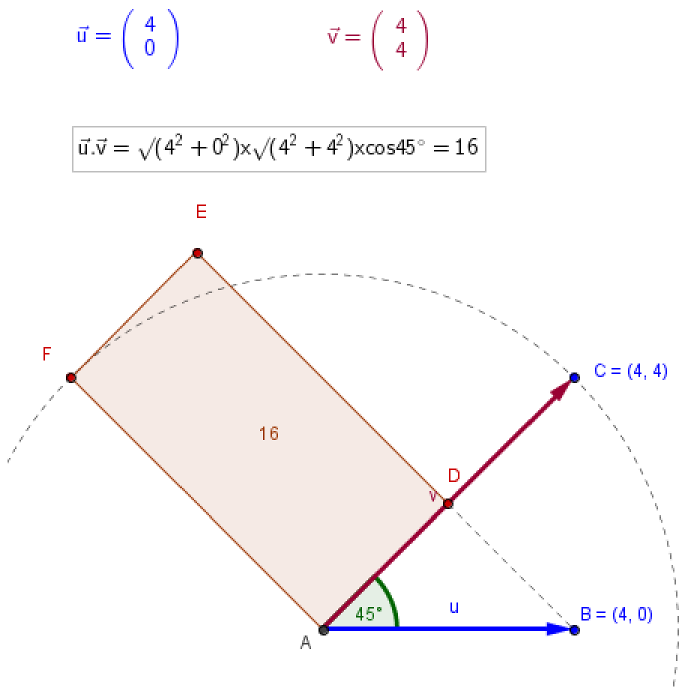

Figure 4: Position of the Applet Discussed by a Student in Lines [9] to [19]

of mathematics [25]. In the moment he uses appropriate mathematical terminology, the *symbolic* representation, i.e. the "S" in the E-I-S model, is achieved. On the next instructor's request ([13]) for justification of this student's explanation based on definition, the student successfully recalls the geometric definition (still written on the board) and derives appropriate argumentation ([14] and [16]). A request for establishing connection between the dot product and the area of the rectangle by the instructor which followed ([17]) did not seem to confuse the student at all. He provided right away correct answer ([18]). Student's argumentation that the area of the rectangle (with sides $|\overrightarrow{u}|$ and $|\overrightarrow{v}_{\overrightarrow{u}}|$ obtained by projection ([10] and [12]) corresponds to dot product of the vectors ([18], "... so therefore it is a Skalarprodukt") actually solves the 'mystery' about what does the resulting scalar represent exactly. Line [18] is actually the **Definition 2** stated in the Subsection 5.2. Thus, the geometric interpretation of the resulting scalar of dot product of two vectors resolves the student's difficulty which closely deals with an important part of the research question, according to the first two guiding features about concept definitions.

The whole above excerpt ([1]-[19]) shows how the student connected arithmetic and geometric mode of description (utilizing cosine of an angle and vector projections) for dot product of vectors with the help of the applet. It seems that the DGE contributed in establishing interconnections between these three mentioned concept definitions, which directly meets guiding features 2 and 3 towards conceptual understanding. Student's interaction with the applet and gestures with his hands rely on Bruner's "E" in the E-I-S theoretical model about what students do and what kind of actions they perform during the learning process. This refers to the Tall's embodied world of mathematics [25]; 2004. Furthermore, student's oral argumentation shows progress in movement from "I" towards "S", thus gaining content-specific knowledge. That is the proceptual world of mathematics in Tall's terminology.

9 Conclusions and Discussion

This study tried to identify five guiding features of conceptual understanding of dot product of vectors. By the help of a designed dynamic geometry environment it investigated how technology could support such identified students' understanding. Namely, advantages provided by technology utilization in the context of the stated research question (RQ) in this study are to:

- help students distinguish what is and what is not a dot product of vectors (guiding feature 1) by providing a dynamic visualization of the resulting scalar;
- facilitate students easily grasp concept definitions and widen concept images (guiding feature 2) by a simultaneous view;
- support students recognize, connect, translate among, manipulate and apply multiple modes of description and thinking (guiding feature 3) by an interactive interchange from one into another mode with a dragging tool;
- enable students connect properties which construct an axiomatic definition (guiding feature 4) by summarizing them into a coherent structure.
- help students connect dot product of vectors with geometry and trigonometry (guiding feature 5).

We briefly summarized the practical implementation of the created dynamic geometry environment, and using the extracts of video recordings, showed that the suggested design supported deeper understanding and contributed to closing the gap between the upper secondary and university approaches in teaching and learning dot product of vectors.

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Software Package

[GeoGebra] http://www.geogebra.org/

Supplemental Electronic Materials

[30] Donevska-Todorova, A. (2012). Interactive Worksheet (on GeoGebraTube) for Investigating Geometric and Algebraic Modes of Description and Scaling Property and Positivity of Dot Product of Vectors.

[31] Donevska-Todorova, A. (2013). Interactive Worksheet (on GeoGebraTube) for Investigating Geometric and Algebraic Modes of Description and Homogenity Property of Dot Product of Vectors.